Solving The Incomplete Markets Model with Aggregate Uncertainty using Explicit Aggregation

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February 2009

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Georgetown Macro Reading Group

Introduction

- Goals: Solve the Krusell-Smith (1998) model using a fast and simple projection algorithm
- Avoid parameterizing the cross-sectional distribution and/or simulating to obtain aggregate laws of motion
- Make clear the relationship between individual state variables and aggregate state variables
- Describe and address the infinite regress problem

Simple Example

- Consider a simple intertemporal optimization problem where all agents are identical except for initial capital
- Parameterize agent i's policy rule:

$$k_i' = \Psi_0(s) + \Psi_1(s)k_i + \Psi_2(s)k_i^2 \tag{1}$$

  where s is a vector of aggregate state variables
- Aggregate (1) to get:

$$\int_0^1 k_i' di = \Psi_0(s) + \Psi_1(s) \int_0^1 k_i di + \Psi_2(s) \int_0^1 k_i^2 di$$
$$\Rightarrow K' = \Psi_0(s) + \Psi_1(s)M(1) + \Psi_2(s)M(2)$$

Simple Example

$$K' = \Psi_0(s) + \Psi_1(s)M(1) + \Psi_2(s)M(2)$$

- $M(1)$ is the aggregate capital stock, and $M(2)$ is the cross-sectional variance
- *Key point:* K' depends on $M(2)$. Therefore, $M(2)$ is a state variable: we will need a law of motion for $M(2)'$
- Next period's cross-sectional variance will enter into next period's individual policy rule

Simple Example

- How to compute $M(2)'$?
- If we had an individual policy rule for $(k_i')^2$, then we could use explicit aggregation
- Suppose we use the individual policy rule implied by (1):

$$(k_i')^2 = \left[\Psi_0(s) + \Psi_1(s)k_i + \Psi_2(s)k_i^2\right]^2 \tag{2}$$

- But then $(k_i')^2$ will depend on k_i^3 and k_i^4
- Aggregating (2) would introduce new aggregate state variables $M(3)$ and $M(4)$
- $\Rightarrow$ *Infinite regress problem*

Simple Example

- Instead, the authors *approximate* $(k_i')^2$ by projecting $(k_i')^2$ on the space of basis functions used to approximate k_i':

$$(k_i')^2 = \Psi_{(k_i')^2,0}(s) + \Psi_{(k_i')^2,1}(s)k_i + \Psi_{(k_i')^2,2}(s)k_i^2 \qquad (3)$$

- Note that this is different from just dropping the coefficients on the third and fourth order terms!
- (3) can now be explicitly aggregated:

$$M(2)' = \Psi_{(k_i')^2,0}(s) + \Psi_{(k_i')^2,1}(s)M(1) + \Psi_{(k_i')^2,2}(s)M(2)$$

- The model can now be solved using a standard projection method

Model to Solve

- Individual agent solves:

$$W(\epsilon, k, a, M; \Psi) = \max_{c,k'} \left\{ u(c) + \beta E \left[W(\epsilon', k', a', M'; \Psi) | \epsilon, a \right] \right\}$$

$$\text{s.t. } k' = (r + 1 - \delta)k + wl - c$$

$$l = \bar{l}\epsilon$$

- Prices r and w are competitive:

$$r = \alpha a \left(\frac{K}{L(a)} \right)^{\alpha - 1}, \; w = (1 - \alpha) a \left(\frac{K}{L(a)} \right)^{\alpha}$$

- $\epsilon \in 0, 1$ is an exogenous, individual employment shock; a is an aggregate productivity shock

Model to Solve

- Let v denote the following error term for an agent:

$$v(\epsilon, k, a, M; \psi) \equiv u'(c) - \beta \sum_{\epsilon', a'} \pi_{aa'\epsilon\epsilon'} u'(c')(r' + 1 - \delta) \quad (4)$$

$$\text{where } c = (r + 1 - \delta)k + wl - k'(\epsilon, k, a, M; \Psi)$$

- Note that v depends on M' indirectly through c'
- FOCs can be written:

$$v(\epsilon, k, a, M; \Psi) \geq 0$$
$$v(\epsilon, k, a, M; \psi)k' = 0$$
$$k' \geq 0$$

- Notice the no-borrowing constraint ($k' \geq 0$)

Solution Approach

Ignoring the borrowing constraint for now, the model can be solved as follows:

1. Construct a grid of the state variables k, M, ϵ, and a
2. Guess a set of coefficients Ψ
3. Given the guess for Ψ, compute the error terms $v(\Psi)$ at each grid point
4. Compute some weighted function of the error terms (e.g., sum of squared errors)
5. If the total weighted error is small enough, then stop; otherwise update Ψ and repeat steps 3 - 5

Solution Approach

- The nontrivial steps in this algorithm are:
  1. Choose the moments that appear in M
  2. Make the necessary approximations to avoid the infinite regress problem
- Authors keep track of aggregate capital for the employed and the unemployed separately – but this is not strictly necessary

Solution Approach

- Individual policy rules:

$$k'_\epsilon = \Psi_{\epsilon,0}(s) + \sum_{i=1}^{I} \Psi_{\epsilon,i}(s) k^i \ , \ \epsilon \in \{u, e\} \tag{5}$$

- Aggregate (5) to get:

$$\hat{K}'_\epsilon = \hat{M}_\epsilon(1)' = \Psi_{\epsilon,0}(s) + \sum_{i=1}^{I} \Psi_{\epsilon,i}(s) M_\epsilon(i) \ , \ \epsilon \in \{u, e\}$$

- Technically, $\hat{K}'_\epsilon$ is end-of-current-period aggregate capital for agents with employment status ϵ. Next period's starting aggregate capital stocks K'_ϵ are simple functions of $\hat{K}'_\epsilon$, a and transition probabilities – see paper for details

Solution Approach

- Let M contain the following moments:

$$M = [M_u(1), \ldots, M_u(I), M_e(1), \ldots, M_e(I)]$$

- *Key step:* Approximate $(k'_\epsilon)^j$ by *projecting* onto the space of the first I monomials of k:

$$(k'_\epsilon)^j = \Psi_{(k'_\epsilon)^j,0}(s) + \sum_{i=1}^{I} \Psi_{(k'_\epsilon)^j,i}(s) k^i \ , \ 1 < j \leq I \qquad (6)$$

- We can then get aggregate laws of motion for $M_\epsilon(j)$, $1 < j \leq I$, by aggregating (6)
- By construction, these aggregate laws of motion will depend only on the elements of M (no infinite regress)

Solution Approach

- In the special case $I = 1$ (linear individual policy rules), the set of aggregate state variables is $M = [M_u(1), M_e(1)] = [K_u, K_e]$, and the extra "infinite-regress projection" is not needed.
- For $I = 2$, explicit calculation of $(k'_\epsilon)^2$ gives:

$$(k'_\epsilon)^2 = \Psi_{\epsilon,0}^2 + 2\Psi_{\epsilon,0}\Psi_{\epsilon,1}k + \left[2\Psi_{\epsilon,0}\Psi_{\epsilon,2} + \Psi_{\epsilon,1}^2\right]k^2 + 2\Psi_{\epsilon,1}\Psi_{\epsilon,2}k^3 + \Psi_{\epsilon,2}^2 k^4 \tag{7}$$

Solution Approach

- One way to carry out the "infinite-regress projection" would be to compute explicit values for $(k'_\epsilon)^2$ at each grid point using (7). Then do OLS of these values on the vector $(k, k^2)'$
- Since k^3 and k^4 are correlated with k and k^2, the projected coefficients capture some of the explanatory power of the higher-order terms. This is more accurate than simply ignoring the higher-order terms, as in Preston and Roca (2007)

Borrowing Constraint

- In order to handle the borrowing constraint, two modifications are necessary:
  1. Allow basis functions to be splines, rather than simple monomials in k:

$$k'_{\epsilon} = \Psi_{\epsilon,0}(s) + \sum_{i=1}^{I} \Psi_{\epsilon,i}(s) B_i(k) \ , \ \epsilon \in \{u, e\}$$

  2. Allow the policy rule used to solve the individual's problem (*individual* policy rule) to differ from the policy rule used for explicit aggregation (*primary auxiliary* policy rule)

Borrowing Constraint

- Author's implementation uses piecewise-linear splines for the individual policy rule, which allows convex savings behavior close to the constraint
- They then use a linear approximation of the individual policy rule to construct the primary auxiliary policy rule
- Since the primary auxiliary policy rule is linear in k, the only aggregate moments needed are K_u and K_e
- Linear approximation introduces a small but systematic bias; this can be removed by increasing the number of basis functions (and state variables) or by implementing a "bias correction" procedure

Comments

- Interesting procedure that highlights the connection between individual and aggregate state variables
- Offers a potential speed improvement over Krussel and Smith (1998), which could be useful for estimation

Comments

- Handling the borrowing constraint complicates the procedure considerably
- Using splines forces either (i) the inclusion of many state variables, or (ii) an artificial distinction between the individual policy rule and the aggregating policy rule
- Even with splines, an extra "bias correction" procedure may be needed to account for the effect of convex savings behavior on the aggregate capital stock
- In the end, the authors use only first moments of the cross-sectional distribution, so much of their projection machinery is not put to use

The End